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On exact correlation functions in SU(N) $\mathcal{N} = 2$ superconformal QCD

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Abstract: We consider the exact coupling constant dependence of extremal correlation functions of $\mathcal{N} = 2$ chiral primary operators in 4d $\mathcal{N} = 2$ superconformal gauge theories with gauge group SU(N) and $N_f = 2N$ massless fundamental hypermultiplets. The 2- and 3-point functions, viewed as functions of the exactly marginal coupling constant and theta angle, obey the tt^* equations. In the case at hand, the tt^* equations form a set of complicated non-linear coupled matrix equations. We point out that there is an ad hoc self-consistent ansatz that reduces this set of partial differential equations to a sequence of decoupled semi-infinite Toda chains, similar to the one encountered previously in the special case of SU(2) gauge group. This ansatz requires a surprising new non-renormalization theorem in $\mathcal{N} = 2$ superconformal field theories. We derive a general 3-loop perturbative formula for 2- and 3-point functions in the $\mathcal{N} = 2$ chiral ring of the SU(N) theory, and in all explicitly computed examples we find agreement with the tt^* equations, as well as the above-mentioned ansatz. This is suggestive evidence for an interesting non-perturbative conjecture about the structure of the $\mathcal{N} = 2$ chiral ring in this class of theories. We discuss several implications of this conjecture. For example, it implies that the holonomy of the vector bundles of chiral primaries over the superconformal manifold is reducible. It also implies that a specific subset of extremal correlation functions can be computed in the SU(N) theory using information solely from the S^4 partition function of the theory obtained by supersymmetric localization.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Gauge Symmetry, Nonperturbative Effects

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1 Introduction

The tt^* equations provide a powerful set of relations between 2- and 3-point functions in the chiral ring of $\mathcal{N} = 2$ theories. They were originally derived with the method of the topological-antitopological fusion in 2d $\mathcal{N} = (2, 2)$ theories in [1]. In 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs) they were derived using superconformal Ward identities in conformal perturbation theory in [2].

There are important differences between $\mathcal{N} = 2$ chiral rings in two and four dimensions that are reflected in the geometry of the superconformal manifold, as well as the structure and solutions of the tt^* equations. For example, 2d $\mathcal{N} = (2, 2)$ chiral rings have a spectrum with an upper bound on the scaling dimension [3]. On the other hand, it is believed that the $\mathcal{N} = 2$ chiral ring of a generic 4d $\mathcal{N} = 2$ SCFT is freely generated without any upper bound on scaling dimensions. As we pointed out in [4] this feature has important implications for the structure of the tt^* equations.

As a more explicit illustration of this fact, in specific 2d theories [1, 5] the tt^* equations take the form of a *periodic* Toda chain. In four-dimensional examples, e.g. the SU(2) $\mathcal{N}=2$ super-Yang-Mills (SYM) theory coupled to 4 hypermultiplets, also known as SU(2) $\mathcal{N}=2$ superconformal QCD (SCQCD), which was analyzed in [4, 6], the tt^* equations also reduce to a Toda chain, but in this case the chain is *semi-infinite*. Periodic and semi-infinite Toda chains have qualitatively different features. The solution of the periodic case is uniquely fixed by unitarity and a small set of perturbative data, but the solution of the semi-infinite case requires complete knowledge of a single member of the chain. The latter does not appear to be uniquely determined by consistency and a few perturbative data. In the context of the SU(2) $\mathcal{N}=2$ SCQCD theory we proposed [4, 6] that the solution can be determined from the exact form of the Zamolodchikov metric on the superconformal manifold, which is the lowest non-trivial member of the Toda chain. In turn, the Zamolodchikov metric, and the exact quantum Kähler potential, are directly related to the S^4 partition function of the theory [7, 8], which can be computed efficiently using localization techniques [9].

The clean example of the SU(2) $\mathcal{N}=2$ SCQCD theory invites us to think more generally about the structure of the tt^* equations in four-dimensional theories, the constraints that they impose on the correlation functions of the $\mathcal{N}=2$ chiral ring, and the independent data needed to determine a physically consistent solution. The precise answer to many of these questions is far from obvious. For instance, already in the general SU(N) SCQCD theory the tt^* equations (in an appropriate gauge) take the form of an infinite set of coupled, non-linear differential equations for matrix-valued quantities whose size grows indefinitely with the scaling dimension (see equation (1.3) below). In more general $\mathcal{N}=2$ SCFTs, which possess higher dimensional superconformal manifolds, the structure of the tt^* equations is an even more complicated set of partial differential equations.

As a step towards a better understanding of this structure, in this paper we initiate a more detailed study of the tt^* equations of the general SU(N) $\mathcal{N}=2$ SCQCD theory. First, by an explicit computation of chiral primary 2- and 3-point functions up to 3-loops in perturbation theory, we verify that the matrix-valued tt^* equations (1.3) are satisfied up to that order. Second, we investigate a specific non-perturbative ansatz for the complete solution of these equations, which is consistent with the perturbative computations. The precise form of this ansatz will be explained in the next subsection. One of its characteristic properties is that it leads to a drastic reduction of the complicated set of matrix-valued equations (1.3) to a decoupled set of semi-infinite Toda chains (similar to the chains encountered in the SU(2) case). These can be solved recursively from a single member in each chain.

Besides this drastic reduction of the tt^* equations the proposed solution has other surprising properties. One of them is the requirement of a novel non-renormalization theorem in this class of $\mathcal{N}=2$ theories, where orthogonal chiral primary operators in a specific basis do not mix by quantum finite coupling effects. Relatedly, the holonomy of the vector bundles of chiral primaries over the superconformal manifold is required to be reducible.

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At the moment, we do not have a proof of the above-mentioned ansatz in gauge theory. Besides the favorable evidence provided by explicit 3-loop computations in perturbation theory, it is encouraging that this ansatz is mathematically a self-consistent way to solve the exact non-perturbative tt^* equations. Nevertheless, we do not have an argument that this is the only way to solve the tt^* equations and the logical possibility of more complicated alternatives (that we have not yet discovered) remains. We point out some alternatives in the main text. A complete non-perturbative proof of the no-mixing conjecture, the explanation of its physical origin, and its relevance in more general $\mathcal{N}=2$ theories are some of the interesting open questions that this work is opening up.

1.1 Summary of main results

The $\mathcal{N}=2$ chiral primary fields of the SU(N) $\mathcal{N}=2$ SYM theory coupled to $2N$ hypermultiplets are believed to be freely generated from the product of a finite set of $N-1$ generators. In the standard Lagrangian description of the theory these generators are represented as single-trace operators $\mathrm{Tr}[\varphi^{\ell+1}]$, $\ell=1,\ldots,N-1$, where φ is the adjoint complex scalar field in the $\mathcal{N}=2$ vector multiplet. From now on, we will denote the generic chiral primary in this representation as ϕ_K with a multi-index $K=\{n_\ell\}$

$$\phi_{\{n_\ell\}} \propto \prod_{\ell=1}^{N-1} \left(\mathrm{Tr}[\varphi^{\ell+1}]\right)^{n_\ell} . \tag{1.1}$$

The anti-chiral primaries are multi-trace operators of the complex-conjugate field $\overline{\varphi}$ and will be denoted as $\overline{\phi}_K$. We single out the special chiral primary $\phi_2 \propto \mathrm{Tr}[\varphi^2]$, which is the single operator with scaling dimension 2 in this family. The supersymmetric descendant $\mathcal{O}_\tau = Q^4 \cdot \phi_2$ of this operator gives the exactly marginal interaction of the theory associated to the complexified coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$. As usual, θ is the theta-angle of the theory and g_{YM} the gauge coupling.

In a specific set of normalization conventions, the so-called holomorphic gauge [4], the OPE coefficients C_{KL}^M are 0 or 1, so that

$$\phi_K(x)\,\phi_L(0) = \phi_{K+L}(0) + \ldots \tag{1.2}$$

Here ϕ_{K+L} denotes the multi-trace operator $:\phi_K\phi_L:$ and the dots represent descendants with higher scaling dimension. In these normalization conventions the tt^* equations become a coupled set of matrix partial differential equations

$$\partial_{\overline{\tau}}\left(g^{\overline{M}_\Delta L_\Delta}\partial_\tau g_{K_\Delta \overline{M}_\Delta}\right) = g_{K_{\Delta+2},\overline{R}_\Delta+\overline{2}}\, g^{\overline{R}_\Delta L_\Delta} - g_{K_\Delta \overline{R}_\Delta}\, g^{\overline{R}_\Delta-\overline{2},L_\Delta-2} - g_2\,\delta_{K_\Delta}^{L_\Delta} \tag{1.3}$$

for the 2-point function coefficients

$$\langle \phi_K(x)\overline{\phi_L}(0)\rangle = \frac{g_{K\overline{L}}}{|x|^{2\Delta}} . \tag{1.4}$$

Here Δ is the common scaling dimension of the insertions. The index 2 in the notation employed in (1.3) refers to the chiral primary ϕ_2. (Further explanations of (1.3) are provided in section 2.)

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In this paper we evaluate the relevant Feynman diagrams and derive a (3-loop) formula that computes $\mathcal{O}(g_{YM}^4)$ corrections to the general 2-point function $g_{K\overline{L}}$ in the SU(N) $\mathcal{N}=2$ chiral ring. We apply this formula in several explicit SU(3) and SU(4) examples and verify that the equations (1.3) are indeed obeyed up to that order.

Moreover, the perturbative results provide highly suggestive evidence that the chiral primary correlators in this class of theories may correspond to a solution of (1.3) of rather special form. The main aim of this paper is to describe the ansatz for this special solution and explore its implications for chiral primary correlators.

Our discussion begins with the special role played by the chiral primary $\phi_2 \propto \mathrm{Tr}[\varphi^2]$. The tt^* equations (1.3) relate the 2-point functions of chiral primaries at a given level of R-charge with those chiral primaries that can be reached by the action of the special chiral primary ϕ_2 or its conjugate $\overline{\phi}_2$. Hence it is natural to consider the operator C_2 (and its conjugate $C_2^\dagger$), corresponding to chiral ring OPE multiplication by ϕ_2, acting on the vector space of chiral primaries. We first show *at tree level* that it is possible to construct a basis of chiral primaries that diagonalizes simultaneously the 2-point functions $g_{K\overline{L}}$ and the action of C_2.[1] Rather surprisingly, we find that even after including 3-loop corrections to these correlators, there still exists a basis where the simultaneous diagonalization of 2-point functions and of C_2 is possible.

This encourages us to investigate the possibility that there is a basis of chiral primaries in which the full non-perturbative matrix of 2-point functions remains diagonal simultaneously with the matrix C_2 for all values of the coupling. This possibility lies at the core of the ansatz that we explore in this paper. The assumption that there is a basis in the chiral ring in which different degenerate operators do not mix under conformal perturbation theory has a simple geometric meaning. It is the geometric statement that the gauge connection on the chiral primary vector bundles over the superconformal manifold is reducible. More specifically, at a generic scaling dimension Δ with degeneracy D the holonomy of the chiral primary vector bundle is, according to this ansatz, not U(D) (as one might have a priori expected), but much smaller, $\mathrm{U}(1)^D$.

As we explain in the main text, the next-to-leading order perturbative results in this paper do not allow us to check conclusively the no-mixing properties for all possible 2-point functions. They only allow us to find direct non-trivial evidence of the absence of mixing for degenerate operators that contain 'a different number of $\mathrm{Tr}\left[\varphi^2\right]$ factors'. This leaves open the possibility of a partial mixing in gauge theory, where at generic scaling dimensions the 2-point functions are non-perturbatively block-diagonal instead of completely diagonal. In that case, the chiral primary vector bundles over the superconformal manifold would be partially reducible to a product of U(1) line bundles times bundles with a non-abelian connection. We explain when non-abelian factors in the holonomy could in principle appear.

Under the postulate of full reducibility the proposed solution implies that the tt^* equations reduce to a *decoupled* set of semi-infinite Toda chains. Each of these Toda chains, whose explicit form can be found in equation (3.20), can be solved in terms of a single external datum. A notable class of data that we can compute in this way are

[1] By this we mean that C_2 sends an element of the basis to a single other element of the basis.

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the 2-point functions of the form $\langle (\mathrm{Tr}[\varphi^2])^n(x)(\mathrm{Tr}[\overline{\varphi}^2])^n(0)\rangle$. Similar to the SU(2) results in [4, 6], we find specific predictions for these data in terms of the Zamolodchikov metric of the SU(N) theory, which is known exactly from supersymmetric localization [8, 9]. We further show that the no-mixing conjecture allows to extract more information from the S^4 partition function for additional extremal correlation functions.

1.2 Outline of the paper

In section 2 we review the basic features of the theory of interest and set up our notation. The precise form of the tt^* equations that we analyze is also reviewed here. Section 3 explains the main proposal and how it leads to a recursive solution of the tt^* equations in the SU(N) theory. Starting at tree level we present a linear transformation on the vector space of chiral primaries that diagonalizes simultaneously the 2-point functions and the components of the OPE coefficients C^L_{2K}, and reorganizes the tt^* equations into a set of decoupled semi-infinite Toda chains. We propose that the nice properties of this basis continue to hold non-perturbatively at finite coupling. The implications of this proposal are discussed further in section 4, which contains a list of specific predictions for exact correlation functions in chiral ring of the SU(N) $\mathcal{N}=2$ SCQCD theory. These predictions are tested non-trivially in perturbation theory in section 5 using a general $\mathcal{O}(g^4_{YM})$ perturbative formula for 2-point functions in the chiral ring derived in appendix A. We conclude in section 6 with a summary of interesting open issues. For the benefit of the reader appendix B contains a supplementary description of the diagonalization of 2-point functions discussed in section 3.

2 Review of the tt^* equations in SU(N) $\mathcal{N}=2$ SCQCD theory

The general properties of the tt^* equations in four-dimensional $\mathcal{N}=2$ SCFTs are reviewed in ref. [4], whose notation we will mostly follow. In the rest of the paper we will omit many of the technical details, which can be found in [4], and will focus directly on the case of interest: the $\mathcal{N}=2$ superconformal QCD (SCQCD) theory defined as $\mathcal{N}=2$ super-Yang-Mills theory with gauge group SU(N) coupled to $2N$ massless hypermultiplets in the fundamental representation. The global symmetry group of this theory for generic N is $\mathrm{U}(2N)\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$, where $\mathrm{U}(2N)$ is the flavor symmetry group and $\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$ is the R-symmetry group.

The $\mathcal{N}=2$ chiral ring of the SU(N) $\mathcal{N}=2$ SCQCD theory is freely generated by the $N-1$ single-trace operators

$$\phi_{\ell+1} \propto \mathrm{Tr}\left[\varphi^{\ell+1}\right], \quad \ell = 1,2,\ldots,N-1 \tag{2.1}$$

with scaling dimension $\Delta=\ell+1$. Here, φ is the adjoint complex scalar field in the $\mathcal{N}=2$ vector multiplet. Hence, the generic chiral primary field

$$\phi_{\{n_\ell\}} \propto \prod_{\ell=1}^{N-1}\left(\mathrm{Tr}[\varphi^{\ell+1}]\right)^{n_\ell} \tag{2.2}$$

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is a multi-trace product of arbitrary powers of these generators. These fields are neutral under the flavor $\mathrm{U}(2N)$ and the R-symmetry group $\mathrm{SU}(2)_R$. They are charged under the $\mathrm{U}(1)_R$ with R-charge R and scaling dimension

$$\Delta_{\{n_\ell\}} = \frac{R_{\{n_\ell\}}}{2} = \sum_{\ell=1}^{N-1} (\ell+1) n_\ell \,. \tag{2.3}$$

The $\mathcal{N}=2$ SCQCD theory has a single (complex) exactly marginal deformation

$$\delta S = \frac{\delta\tau}{4\pi^2} \int d^4x \, \mathcal{O}_\tau(x) + \frac{\delta\overline{\tau}}{4\pi^2} \int d^4x \, \overline{\mathcal{O}}_\tau(x) \,, \tag{2.4}$$

where $\mathcal{O}_\tau$ is the supersymmetry descendant

$$\mathcal{O}_\tau = Q^4 \cdot \phi_2 \tag{2.5}$$

of the chiral primary $\phi_2 \propto \mathrm{Tr}[\varphi^2]$. The notation $Q^4 \cdot \phi_2$ denotes the nested (anti)-commutator of four supercharges of left chirality on the field ϕ_2. The corresponding exactly marginal coupling that parametrizes the (complex) 1-dimensional superconformal manifold is the complexified gauge coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{YM}}$.

In what follows we will employ a specific set of normalization conventions for the chiral primaries $\phi_{\{n_\ell\}}$, where (i) ϕ_2 adheres to the conventions (2.4)–(2.5), (ii) we require

$$\langle \mathcal{O}_\tau(x) \overline{\mathcal{O}}_\tau(0) \rangle = \nabla^2_x \nabla^2_x \langle \phi_2(x) \overline{\phi}_2(0) \rangle \,, \tag{2.6}$$

(iii) the remaining generators in (2.1) are chosen with an arbitrary holomorphic normalization factor, and (iv) we require that the non-vanishing OPE coefficients are

$$C_{K\,L}^{K+L} = 1 \,. \tag{2.7}$$

For general indices $K, L, \ldots$ of the form $\{k_\ell\}, \{l_\ell\}, \ldots$ the notation $K+L$ in (2.7) denotes the index $\{k_\ell + l_\ell\}$ and the OPE coefficient (2.7) implies the operator product expansion

$$\phi_{\{k_\ell\}}(x) \, \phi_{\{l_\ell\}}(0) = \phi_{\{k_\ell + l_\ell\}}(0) + \cdots \,. \tag{2.8}$$

This OPE is enough to fix the normalization of all multi-trace chiral primaries in terms of the normalization of the generators (2.1).

As explained in [2, 4] it is most appropriate to think of the chiral primary fields ϕ_L as sections in a holomorphic vector bundle $\mathcal{V}$ whose base space is the superconformal manifold of the theory. The above set of conventions is a choice that makes the rescaled chiral primary fields

$$e^{-\frac{R_L}{c'}\mathcal{K}} \phi_L \tag{2.9}$$

holomorphic sections of the bundle $\mathcal{V}$. Here, R_L is the $\mathrm{U}(1)_R$ charge of the fields ϕ_L, $c' = 8 \times 192 \times c$ (where c is the central charge of the CFT), and $\mathcal{K}$ is the exact Kähler potential of the superconformal manifold. ϕ_L are the chiral primaries in the conventions (2.6)–(2.7). The reason for the appearance of the factor $e^{-\frac{R_L}{c'}\mathcal{K}}$ can be traced back to the choice of

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normalization conventions for the supercharges, or equivalently to the choice of a section in the holomorphic line bundle associated to the supercurrents.

These choices constitute the so-called holomorphic gauge where a connection A on $\mathcal{V}$ compatible with the 2-point function coefficients $g_{K\overline{L}}$ has components [2]

$$(A_\tau)_K^L = g^{\overline{M}L}\partial_\tau g_{K\overline{M}} - \frac{2R_K}{c'}\partial_\tau \mathcal{K}\,\delta_K^L\,, \tag{2.10}$$

$$(\overline{A}_{\overline{\tau}})_{\overline{K}}^{\overline{L}} = g^{\overline{L}M}\partial_{\overline{\tau}} g_{M\overline{K}} - \frac{2R_{\overline{K}}}{c'}\partial_{\overline{\tau}} \mathcal{K}\,\delta_{\overline{K}}^{\overline{L}}\,. \tag{2.11}$$

We remind the reader that $g_{K\overline{L}}$ is defined in (1.4). The notation $g^{\overline{K}L}$ refers to the components of the inverse matrix of 2-point function coefficients: $g_{K\overline{M}}g^{\overline{M}L} = \delta_K^L$. The components $(A_\tau)_{\overline{L}}^{\overline{K}}$, $(\overline{A}_{\overline{\tau}})_L^K$ vanish by definition in the holomorphic gauge.

The tt^* equations express the curvature of this connection. In holomorphic gauge they lead to a set of partial differential equations for the 2- and 3-point function coefficients $g_{K\overline{L}}, C_{KL}^M$, which have the form

$$\begin{aligned}\partial_{\overline{\tau}}&\left(g^{\overline{M}_\Delta L_\Delta}\partial_\tau g_{K_\Delta \overline{M}_\Delta}\right) \\ &= C_{2K_\Delta}^{P_{\Delta+2}} g_{P_{\Delta+2}\overline{Q}_{\Delta+2}}\, C_{\overline{2R}_\Delta}^{*\overline{Q}_{\Delta+2}} g^{\overline{R}_\Delta L_\Delta} - g_{K_\Delta \overline{N}_\Delta}\, C_{\overline{2U}_{\Delta-2}}^{*\overline{N}_\Delta} g^{\overline{U}_{\Delta-2}V_{\Delta-2}} C_{2V_{\Delta-2}}^{L_\Delta} - g_2\,\delta_{K_\Delta}^{L_\Delta}\,,\end{aligned} \tag{2.12}$$

where C_{2K}^L denotes the coefficient in the OPE of the chiral primaries ϕ_2 and ϕ_K. In the conventions (2.7) we set $C_{2K}^L = \delta_{K+2}^L$ and (2.12) simplifies to (1.3). Note that in (2.12) we are using the more explicit index notation $K_\Delta, \ldots$ to keep track of the scaling dimension Δ of the corresponding chiral primary fields. In this way it is apparent that equation (2.12) is an equation that relates 2-point functions at three different scaling dimensions: $\Delta - 2$, Δ, and $\Delta + 2$.

Finally, $g_2 = \langle \mathrm{Tr}[\varphi^2](1)\mathrm{Tr}[\overline{\varphi}^2](0)\rangle$ is related by (2.6) to the Zamolodchikov metric of the theory up to an overall constant factor. Hence g_2 is conveniently related [7] to the S^4 partition function of the theory, Z_{S^4}, which is exactly computable as an $(N-1)$-dimensional ordinary integral with the use of supersymmetric localization methods [9]. The precise relation between g_2 and Z_{S^4} in our conventions is

$$g_2 = \partial_\tau \partial_{\overline{\tau}} \log Z_{S^4}\,. \tag{2.13}$$

This equation was first proven in [7].

In the special case of the SU(2) theory, which was the focus of ref. [6], the chiral ring is freely generated by $\mathrm{Tr}[\varphi^2]$ only, and then (2.7), (2.12) reduce to the simple recursive set of differential equations

$$\partial_{\overline{\tau}}\partial_\tau \log g_{2n} = \frac{g_{2n+2}}{g_{2n}} - \frac{g_{2n}}{g_{2n-2}} - g_2\,, \quad n = 1, \ldots \tag{2.14}$$

where by definition $g_0 = 1$. These equations can be recast as a semi-infinite Toda chain

$$\partial_{\overline{\tau}}\partial_\tau q_n = e^{q_{n+1}-q_n} - e^{q_n - q_{n-1}}\,, \quad n = 1, \ldots \tag{2.15}$$

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by setting

$$g_{2n} \equiv \langle \left(\mathrm{Tr}[\varphi^2]\right)^n (1) \left(\mathrm{Tr}[\overline{\varphi}^2]\right)^n (0)\rangle = \exp\left(q_n - \log Z_{S^4}\right) . \tag{2.16}$$

In the general SU(N) case, due to the presence of additional chiral ring generators (2.1), the equations (2.12) are instead a complicated set of coupled differential equations for matrix-valued quantities. The appearance of inverse matrices introduces a high level of non-linearity.

In [6] we provided an explicit independent check that the matrix-valued equations (2.12) are satisfied at tree level for any SU(N) $\mathcal{N}=2$ SCQCD theory. In this work we provide additional non-trivial independent evidence for the validity of the tt^* equations, by computing the first quantum corrections which arise at 3-loops in perturbation theory.

It is clear that a single datum, like the S^4 partition function, is not enough to obtain a full solution of the tt^* equations in the SU(N) theory with $N > 2$. For example, since the scaling dimensions of the fields appearing in (2.12) are related by an increment of 2, the equations for chiral primaries of even scaling dimension are decoupled from the equations of the chiral primaries of odd scaling dimension. Hence, separate data are needed to solve the tt^* equations for the chiral primaries of odd scaling dimension. Moreover, even within the sector of even or odd scaling dimensions, the pattern of increasing degeneracies does not admit an obvious recursive solution as in the simple SU(2) case. In what follows, we propose a surprising reduction of this problem.

3 Decoupling the tt^* equations

In this section we set up an ansatz that decouples the tt^* equations and allows us to solve them recursively as a set of independent semi-infinite Toda chains. First, we examine the structure of the chiral ring at tree level. We focus on two natural operators defined on the space of chiral primaries, which correspond to the action by the OPE with the $\Delta = 2$ chiral primary $\phi_2 = \mathrm{Tr}[\varphi^2]$ and the anti-chiral primary $\overline{\phi}_2 = \mathrm{Tr}[\overline{\varphi}^2]$. We show that these two operators, called C_2 and $C_2^\dagger$, are the adjoint of one another and satisfy a simple algebra according to which they can be treated as creation and annihilation operators. This allow us to decompose the space of chiral primaries into representations of this algebra, which are orthogonal with respect to the 2-point functions. In this basis the tree level tt^* equations explicitly decouple into a set of independent semi-infinite Toda chains.

At a second stage, we examine how this structure is modified at finite coupling. We argue that the tt^* equations (2.12) — seen abstractly as a set of coupled differential equations — admit a class of solutions, in which different chiral primaries in the above basis remain orthogonal for all values of the coupling. This class does not necessarily include the most general solution of the tt^* equations and a priori it is not clear whether it includes the physically relevant solution that we seek in the context of the gauge theory. Perturbative evidence in favor of the physical relevance of this restricted class is provided in section 5.

3.1 The chiral ring at tree level

The basis of chiral primaries generated by the single-trace operators (2.1) is particularly convenient because the structure constants are very simple — see equation (2.7). Nev-

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ertheless, it is straightforward to check that the 2-point functions in this basis are *not* diagonal. In turn, this means that the tt^* equations in (2.12) do not reduce to simple recursive one-dimensional chains of equations, but rather they constitute a set of coupled non-linear partial differential equations where various components of the 2-point functions mix nontrivially among themselves. As a result, it makes sense to look for a new basis that diagonalizes the 2-point functions while preserving some of the simplicity of (2.7). Since the only structure constants that appear in the tt^* equations are the ones that involve ϕ_2, namely C_{2K}^L, it will suffice to look for a basis where these structure constants remain diagonal, i.e. as matrices they have a single non-vanishing element at each row.

Along these lines let us consider first chiral primaries $\phi_K^{(0)}$ with the defining OPE property

$$\overline{\phi_2}(x)\phi_K^{(0)}(0) = \frac{0}{|x|^4} + \dots . \tag{3.1}$$

These are chiral primaries where the most singular term in the OPE with $\overline{\phi_2}$ vanishes.[2] Henceforth, we will refer to these distinguished chiral primaries as "C_2-primaries".

We can construct generic chiral primaries by acting repeatedly on the C_2-primaries with $\phi_2 \propto \mathrm{Tr}\left[\varphi^2\right]$

$$\phi_K^{(n)} \equiv \phi_2^n \phi_K^{(0)} . \tag{3.2}$$

As an obvious benefit, the structure constants C_{2K}^L are manifestly diagonal in this basis. Hence, for our purposes we would only need to show that the 2-point functions at tree level are diagonal as well. For starters, let us show that

$$\langle \phi_K^{(m)}(x)\, \overline{\phi_L^{(n)}}(0)\rangle = 0\,, \quad \text{if } m \neq n\,. \tag{3.3}$$

In any basis, $\mathcal{N}=2$ chiral primaries exhibit the OPEs

$$\phi_2(x)\phi_K(0) = C_{2K}^L \phi_L(0) + \dots , \tag{3.4}$$

$$\overline{\phi_2}(x)\phi_K(0) = g_{K\overline{R}} C_{\overline{2P}}^{*\overline{R}} g^{\overline{P}L} \phi_L(0) \frac{1}{|x|^4} + \dots . \tag{3.5}$$

Hence, as we implied already, there are two natural operators acting on the space of chiral primaries

$$(C_2)_K^L \equiv C_{2K}^L\,, \tag{3.6}$$

$$\left(C_2^\dagger\right)_K^L \equiv g_{K\overline{R}} C_{\overline{2P}}^{*\overline{R}} g^{\overline{P}L} . \tag{3.7}$$

Put differently, if we have a chiral primary $\phi = v^K \phi_K$, where v^K is an arbitrary vector, then $C_2 \cdot \phi$ is the chiral primary that appears in the OPE of ϕ with ϕ_2, and $C_2^\dagger \cdot \phi$ is the chiral primary that appears in the OPE of ϕ with $\overline{\phi_2}$, that is

$$C_2 \cdot \phi \equiv v^K C_{2K}^L \phi_L\,, \tag{3.8}$$

$$C_2^\dagger \cdot \phi \equiv v^K g_{K\overline{R}} C_{\overline{2P}}^{*\overline{R}} g^{\overline{P}L} \phi_L . \tag{3.9}$$

[2]It is easy to see using $\mathrm{U}(1)_R$ conservation and the unitarity bound $\Delta \geq \frac{|R|}{2}$ that, as long as $\Delta_K \geq 2$, the most singular term on the r.h.s. of (3.1) is of the form $\frac{\phi(0)}{|x|^4}$ where $\phi(0)$ is a chiral primary. Of course, in specific cases, such as the C_2-primaries that we define above, this term may be absent from the OPE.

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In particular, the operator C_2 raises the R-charge of an operator by 4 and the conformal dimension by 2, while $C_2^\dagger$ lowers them by the same amount.

We will now show that $C_2^\dagger$ is the adjoint of C_2 with respect to the metric defined by the 2-point functions. Consider the 3-point function

$$\langle \phi(x_1)\overline{\phi_2}(x_2)\overline{\phi'}(x_3)\rangle = \frac{\alpha}{|x_{12}|^4|x_{13}|^{2\Delta-4}}\,, \tag{3.10}$$

where α is a constant and we used $\Delta = \Delta' + 2$ due to R-charge conservation. In the limit $x_2 \to x_3 \to 0$, $x_1 \to 1$ we find

$$\alpha = \langle \phi(1)\,\overline{C_2 \cdot \phi'}(0)\rangle\,, \tag{3.11}$$

while in the limit $x_1 \to x_2 \to 1$, $x_3 \to 0$ we find

$$\alpha = \langle C_2^\dagger \cdot \phi(1)\,\overline{\phi'}(0)\rangle\,. \tag{3.12}$$

The combination of equations (3.11), (3.12) verifies the advertised statement: $C_2^\dagger$ is indeed the adjoint of C_2 with respect to the metric induced by the 2-point functions.

There is a second crucial property of the operators C_2, $C_2^\dagger$. Their commutator acts as follows

$$[C_2, C_2^\dagger]\cdot\phi = v^K\left(g_{K\overline{R}}C^{*\overline{R}}_{\overline{2}\overline{P}}g^{\overline{P}L}C^Q_{2L} - C^L_{2K}g_{L\overline{R}}C^{*\overline{R}}_{\overline{2}\overline{P}}g^{\overline{P}Q}\right)\phi_Q \tag{3.13}$$

$$\equiv -v^K[C_2,\overline{C_2}]^Q_K\,\phi_Q\,. \tag{3.14}$$

The combination $[C_2,\overline{C_2}]^Q_K$ satisfies at tree level a nice combinatorial identity that was proven in appendix C of [4]

$$[C_2,\overline{C_2}]^L_K = g_{2\overline{2}}\delta^L_K\left(1+\frac{R}{\dim\mathcal{G}}\right)\,. \tag{3.15}$$

Hence, when we plug this identity into (3.13), we find

$$[C_2, C_2^\dagger]\cdot\phi = -g_{2\overline{2}}\left(1+\frac{R}{\dim\mathcal{G}}\right)\phi\,. \tag{3.16}$$

This means that we can regard C_2 and $C_2^\dagger$ as creation and annihilation operators respectively. In other words, we can decompose the space of chiral primaries in terms of representations of this algebra. We start from 'highest weight chiral primaries' annihilated by $C_2^\dagger$ (the C_2-primaries), and we build the space of states by acting with C_2 (multiplying by ϕ_2, as in (3.2)). Then, the resulting representations must necessarily be orthogonal, i.e. they diagonalize the 2-point functions. At this stage this is easy to verify directly if, say, $m < n$ and $C_2^\dagger\cdot\phi^{(0)} = 0$. Indeed,

$$\langle C_2^m\cdot\phi^{(0)}(x)\,\overline{C_2^n\cdot\phi'}(0)\rangle = \langle (C_2^\dagger)^n C_2^m\cdot\phi^{(0)}(x)\,\overline{\phi'}(0)\rangle = 0\,, \tag{3.17}$$

where in the last step we used repeatedly the commutator $[C_2, C_2^\dagger]$.

In the special case $m = n$, i.e. when two chiral primaries ϕ, ϕ' are degenerate, we obtain similarly the identity

$$\langle C_2^m\cdot\phi(x)\,\overline{C_2^m\cdot\phi'}(0)\rangle \propto g^m_{2\overline{2}}\langle\phi(x)\overline{\phi'}(0)\rangle\,. \tag{3.18}$$

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Consequently, if there is a degeneracy in the spectrum of C_2-primaries, we can choose any orthogonal combination and the orthogonality will be preserved by the C_2-descendants. As a result, the basis built on C_2-primaries that is singled out in this section is not unique. At tree level any orthogonal combination of degenerate C_2-primaries is equally acceptable for our purposes. It is nevertheless useful to keep this freedom in mind in the context of finite coupling effects, which will be discussed shortly.

3.2 tt^* equations at tree level

It is rather straightforward to show that, in the basis defined in the previous section, the tree level tt^* equations decouple into a collection of one-dimensional semi-infinite Toda chains. Indeed, starting with a given arbitrary C_2-primary $\phi^{(0)}$, let us consider the subsequence of chiral primaries $\phi^{(n)} = \phi_2^n \phi^{(0)}$. Since the 2-point functions are diagonal at tree level, we can focus on the components

$$G_{2n} \equiv \langle \phi^{(n)}(1)\overline{\phi^{(n)}}(0)\rangle \,. \tag{3.19}$$

Inserting the results of the previous subsection into the tree level version of (2.12), it is easy to see that the G_{2n} satisfy

$$\partial_{\overline{\tau}}\partial_\tau \log G_{2n} = \frac{G_{2n+2}}{G_{2n}} - \frac{G_{2n}}{G_{2n-2}} - g_2 \,, \tag{3.20}$$

which is very similar to the one-dimensional chain (2.14) of the SU(2) case (and can be recast as the semi-infinite Toda chain (2.15)).

At tree level it is not hard to solve (3.20) explicitly in closed form. Let us assume that the C_2-primary $\phi^{(0)}$ has scaling dimension Δ_0. Then, the generic chiral primary $\phi^{(n)} = \phi_2^n \phi^{(0)}$ has scaling dimension $\Delta_0 + 2n$, and at tree level

$$G_{2n} = \frac{1}{(\mathrm{Im}\tau)^{\Delta_0+2n}} \tilde{G}_{2n} \,, \tag{3.21}$$

where $\tilde{G}_{2n}$ are $(\tau, \overline{\tau})$-independent constants determined solely by group-theoretical Wick contractions. Implementing (3.21) equation (3.20) becomes

$$\frac{\Delta_0 + 2n}{4} = \frac{\tilde{G}_{2n+2}}{\tilde{G}_{2n}} - \frac{\tilde{G}_{2n}}{\tilde{G}_{2n-2}} - \tilde{g}_2 \,. \tag{3.22}$$

Moreover, in our conventions

$$\tilde{g}_2 = \frac{N^2-1}{8} \,. \tag{3.23}$$

Consequently, solving (3.22) we obtain

$$\tilde{G}_{2n} = \frac{\tilde{G}_0}{4^n} \prod_{\ell=0}^{n-1} \left[\frac{4\tilde{G}_2}{\tilde{G}_0} + \ell \left(\frac{N^2-1}{2} + \Delta_0 + 1 \right) + \ell^2 \right] \tag{3.24}$$

in terms of the numerical 2-point function coefficients $\tilde{G}_0$, $\tilde{G}_2$ for the correlators $\langle \phi^{(n)}(1)\overline{\phi^{(n)}}(0)\rangle$ with $n = 0, 1$.

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In the special case, where $\phi^{(0)} = \mathbf{1}$ (the identity operator), the sequence $\phi^{(n)} = \phi_2^n$ is comprised of the same type of operators that constitute the $\mathcal{N} = 2$ chiral ring in the SU(2) case. For those operators we have $G_{2n} = g_{2n}$, and equation (3.20) is exactly the same semi-infinite Toda chain that was encountered in the SU(2) case (2.14). In this situation the solution (3.24) simplifies to

$$\tilde{g}_{2n} = \frac{n!}{4^n}\left(\frac{N^2-1}{2}\right)_n \,, \tag{3.25}$$

where $(x)_n$ is the Pochhammer symbol

$$(x)_n = x(x+1)\cdots(x+n-1)\,. \tag{3.26}$$

This relation was noticed empirically and conjectured to hold for the general SU(N) theory in [4]. Amusingly, a similar relation has been proven with direct methods some time ago in appendix A.4 of ref. [10] for the U(N) $\mathcal{N} = 4$ SYM theory. The SU(N) and U(N) formulae are identical with the suggestive substitution of $N^2 - 1$ with N^2 (the dimension of the gauge group) inside the Pochhammer symbol (3.25). The formula (3.24) is an interesting generalization to arbitrary $\mathcal{N} = 2$ chiral primary operators. It is equally applicable to chiral primary operators in SU(N) $\mathcal{N} = 4$ SYM theory, where the tree level 2- and 3-point functions do not receive quantum corrections.

3.3 tt^* equations at finite coupling: a no-mixing ansatz

Having shown at tree level that the tt^* equations decouple into a sequence of independent Toda chains, it is natural to ask if a similar decoupling continues to hold when quantum corrections are taken into account. A sufficient condition for this effect is the requirement that the full non-perturbative 2-point functions remain diagonal in at least one of the bases constructed in the previous subsections, call it $\hat{\phi}_K$ (recall that the previous tree level construction of the bases based on C_2-primaries was not unique). This requirement is mathematically consistent from the point of view of the tt^* equations. Notice that what we are postulating here is essentially the ability to diagonalise the exact 2-point functions *within the holomorphic gauge.*

We would like to stress that the 2-point functions can always be diagonalized in a suitable basis. However, such a basis will generically be non-holomorphic and it is not clear how one would obtain a solution to the resulting tt^* equations. Having said this, we should keep in mind that our postulate does not exclude the possibility of holomorphic corrections to the tree-level basis.

From the point of view of the gauge theory the no-mixing condition postulated by the above ansatz appears to be a new non-renormalization theorem in a four-dimensional $\mathcal{N} = 2$ theory. Notice that unlike the non-renormalization theorem in $\mathcal{N} = 4$ SYM [11–20], this theorem would not fix completely the moduli-dependence of correlation functions in the chiral ring. So far we have not been able to prove it using superconformal Ward identities. If true, this theorem would lead to several non-trivial consequences, which are discussed in detail in the next section. For instance, it would imply geometrically that the gauge

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connection of the holomorphic chiral primary vector bundles on the $\mathcal{N}=2$ superconformal manifold are reducible.

In appendix B we also formulate the no-mixing condition in terms of the original multi-trace basis ϕ_K of equation (2.2). In that basis, the non-renormalization condition translates into a statement about the coupling constant independence of appropriate ratios of 2-point functions.

In section 5 we put the above ansatz to the test in perturbation theory by computing the first non-trivial quantum corrections to several 2-point functions of chiral primary operators in SU(3) and SU(4) SCQCD. We proceed as high in scaling dimension and gauge group rank as possible, given our current computational limitations with the complicated combinatoric structures at 3-loops. In all cases, we verify the no-mixing ansatz: the 2-point functions remain diagonal, and the decoupled Toda equations (3.20) are explicitly verified. This evidence seems to be highly suggestive. Since the no-mixing ansatz provides a consistent non-trivial recursive solution to the tt^* equations non-perturbatively, it is natural to anticipate that it will hold beyond the perturbative results of section 5.

4 Implications of the decoupling

In this section, we explore some of the implications of the conjectured no-mixing condition and the related decoupling of the tt^* equations. First, we show that, similar to the SU(2) case [4], we can use the decoupled Toda chains to determine exact correlation functions in the $\mathcal{N}=2$ chiral ring of the SU(N) theory from a single datum in each decoupled subsector. In particular, we show that the subsequence based on the identity operator is exactly solvable using current knowledge from supersymmetric localization. With the same data we also obtain predictions for the exact form of certain extremal correlators that involve only single-trace operators, e.g. certain single-trace 3-point functions.

Second, we examine the geometric interpretation of this decoupling and show that it implies that the holonomy group on the space of chiral primaries is restricted — assuming full decoupling the holonomy group is a product of abelian factors.

Finally, we present other implications of the no-mixing ansatz on general (not necessarily extremal) integrated correlation functions.

4.1 Proposed recursive solution of the tt^* equations at finite coupling

We begin by rewriting (3.20) in the recursive form

$$G_{2n+2} = G_{2n}\partial_\tau \partial_{\bar{\tau}} \log G_{2n} + \frac{G_{2n}^2}{G_{2n-2}} + G_{2n} g_2 \,. \tag{4.1}$$

Assuming the no-mixing ansatz of subsection 3.3 this is now an equation that holds non-perturbatively in the SU(N) theory at finite coupling. The solution is determined recursively from the 2-point function of the C_2-primary operator $\phi^{(0)}$ under consideration

$$G_0 = \langle \phi^{(0)}(1)\, \overline{\phi^{(0)}}(0)\rangle \tag{4.2}$$

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and the 2-point function of the unique $\Delta = 2$ chiral primary ϕ_2

$$g_2 = \langle \phi_2(1)\, \overline{\phi_2}(0) \rangle \,, \tag{4.3}$$

which is, up to a convention-dependent numerical coefficient, the Zamolodchikov metric. Different choices of the operator $\phi^{(0)}$ sample different subsectors of the $\mathcal{N} = 2$ chiral ring and correspond to different solutions of the recursive equations (4.1).

As we noted already in section 2, the Zamolodchikov metric can be obtained from the S^4 partition function using supersymmetric localization [7, 9]. In the SU(N) $\mathcal{N} = 2$ SCQCD theory the S^4 partition function can be written as an $(N-1)$-dimensional ordinary integral. We refer the reader to the original references for explicit formulae.

To the best of our knowledge, it is not currently known how to compute the general G_0 exactly as a function of the moduli for arbitrary C_2-primaries. A notable exception is the main subsequence defined by the identity operator, $\phi^{(0)} = \mathbf{1}$. As we pointed out already in subsection 3.2, in this case the 2-point functions $G_{2n} \equiv g_{2n}$ satisfy equation (2.14), so the analysis of [4, 6] can be repeated almost without changes. The only difference is that the starting point g_2 must be computed from the S^4 partition function for the gauge group SU(N) instead of SU(2).

4.2 Single-trace extremal correlation functions and large-N limits

According to our conjecture, current knowledge of the Zamolodchikov metric also gives exact predictions for certain extremal correlation functions that involve only *single-trace* chiral primaries. Correlation functions of single-trace operators have an obvious interest in large-N limits.

As an illustrating example, let us consider first such a 3-point function of single-trace operators in the SU(4) theory. It will be shown in the next section that the only C_2-primary at scaling dimension $\Delta = 4$ is

$$\phi_4^{(0)} = \mathrm{Tr}[\varphi^4] - \frac{29}{68}\mathrm{Tr}[\varphi^2]^2 \,. \tag{4.4}$$

Our conjecture implies in particular that

$$\langle \phi_2(\infty)\phi_2(1)\overline{\phi_4^{(0)}}(0)\rangle = 0 \,. \tag{4.5}$$

Moreover, we know that

$$\langle \phi_2(\infty)\phi_2(1)\overline{\phi_2^2}(0)\rangle = C_{\phi_2\phi_2}^{\phi_2^2}\langle \phi_2^2(1)\overline{\phi_2^2}(0)\rangle = g_4 \,, \tag{4.6}$$

with g_4 being determined from g_2 and the Toda equation as

$$g_4 = g_2\, \partial_{\bar\tau}\partial_\tau \log g_2 + 2\, g_2^2 \,. \tag{4.7}$$

Together with equation (4.5) we deduce that

$$\langle \mathrm{Tr}[\varphi^2](\infty)\, \mathrm{Tr}[\varphi^2](1)\, \mathrm{Tr}[\overline{\varphi}^4](0)\rangle = \frac{29}{68} g_4 \,. \tag{4.8}$$

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Notice that (4.8) is an exact expression for a 3-point function in a specific set of normalization conventions, where the 2-point functions $\langle \mathrm{Tr}[\varphi^2](1)\,\mathrm{Tr}[\overline{\varphi}^2](0)\rangle$, $\langle \mathrm{Tr}[\varphi^4](1)\,\mathrm{Tr}[\overline{\varphi}^4](0)\rangle$ are fixed. Although it is known at the moment how to compute exactly the first of these 2-point functions using localization, it is not known how to compute the second.

The result (4.8) has the following straightforward generalization. From the analysis of ref. [4] we can deduce, using superconformal Ward identities, the identity

$$\langle \mathrm{Tr}[\varphi^2](x_1)\mathrm{Tr}[\varphi^2](x_2)\cdots\mathrm{Tr}[\varphi^2](x_k)\mathrm{Tr}[\overline{\varphi}^{2k}](\infty)\rangle = \langle \mathrm{Tr}[\varphi^2]^k(0)\mathrm{Tr}[\varphi^{2k}](\infty)\rangle\,. \tag{4.9}$$

The extremal $(k+1)$-point function in question is independent of the insertions of the operators and equal to a 2-point function for two operators at scaling dimension $2k$. Following an argument similar to the one of the previous paragraphs, or equivalently the non-renormalization identities of appendix B, we obtain

$$\langle \mathrm{Tr}[\varphi^2]^k(0)\mathrm{Tr}[\varphi^{2k}](\infty)\rangle = \frac{\langle \mathrm{Tr}[\varphi^2]^k(0)\mathrm{Tr}[\varphi^{2k}](\infty)\rangle_{\mathrm{tree}}}{\langle \mathrm{Tr}[\varphi^2]^k(0)\mathrm{Tr}[\varphi^2]^k(\infty)\rangle_{\mathrm{tree}}}\, g_{2n}\,, \tag{4.10}$$

where the prefactor is evaluated at tree level, and g_{2n} is determined from the S^4 partition function and the chain (2.14).

It would be interesting to study the behavior of such single-trace correlators further in the large-N limit and explore possible implications in related applications of the AdS/CFT correspondence. We note in passing that, as a simple check of our formalism and the tt^* equations (2.12) in the large-N limit, one can easily verify that large-N factorization is an automatic solution of the tt^* equations.

More along the lines of the large-N limit, the recent work [21] studied in supergravity the structure of the moduli space of certain supersymmetric AdS_5 vacua, which have the right amount of supersymmetry to be the holographic duals of 4d $\mathcal{N}=2$ SCFTs. If these theories have a holographic dual, then the moduli space of vacua in supergravity correspond to the conformal manifold of the dual SCFT. It would be interesting to investigate the form of the Zamolodchikov metric in the large-N limit directly from the gauge theory and to compare it with the supergravity results of [21].

4.3 Reducible chiral primary bundles

Another consequence of the existence of a holomorphic basis $\hat{\phi}_K$ that diagonalizes the 2-point functions non-perturbatively is that the vector bundles of chiral primaries are *reducible*. It is readily seen from equations (2.10)–(2.11), that the connection $\hat{A}$ is diagonal in the basis of the operators $\hat{\phi}_K$. Consequently, if there are D chiral primaries of scaling dimension Δ (at arbitrary Δ), the holonomy will be restricted to the subgroup $\mathrm{U}(1)^D \subset \mathrm{U}(D)$. A reducible holonomy is a non-trivial condition for the geometry of the chiral primary vector bundles over the superconformal manifold.

The strong version of the no-mixing conjecture proposed in subsection 3.3 states that the 2-point functions are fully diagonalizable non-perturbatively in a holomorphic basis, hence the connection and the associated holonomy are fully reducible. Notice that full reducibility is consistent with the operator product structure

$$C\ :\ \mathcal{V}_\Delta \times \mathcal{V}_{\Delta'}\ \to\ \mathcal{V}_{\Delta+\Delta'} \tag{4.11}$$

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that allows us to multiply sections from two chiral primary vector bundles to obtain a section on a third chiral primary vector bundle at the sum of scaling dimensions.

Currently we have not excluded the possibility of a consistent weaker version of the no-mixing conjecture, where the holonomy is partially reducible to a subgroup that is a product of abelian and non-abelian factors. In the next section, where we provide direct evidence for decoupling in perturbation theory, we verify that 2-point functions

$$\langle \phi^{(m)}(x)\, \overline{\phi^{(n)}}(0) \rangle = 0\,, \quad \text{with } m \neq n \tag{4.12}$$

do not mix at the quantum level. In all the cases that we have analyzed so far, the degenerate operators are C_2-descendants of primaries at different scaling dimensions. Interesting subtleties, with potential non-abelian holonomies, could seemingly appear in situations with more than one degenerate C_2-primary operators. Recall that this was precisely the origin of the non-uniqueness of the basis constructed from the C_2-algebra in section 3.1.

For example, if $N \geq 6$, the $\Delta = 6$ spectrum includes the operators

$$\mathrm{Tr}\left[\varphi^6\right]\,, \quad \mathrm{Tr}\left[\varphi^3\right]^2\,, \quad \mathrm{Tr}\left[\varphi^4\right]\mathrm{Tr}\left[\varphi^2\right]\,, \quad \mathrm{Tr}\left[\varphi^2\right]^3\,. \tag{4.13}$$

It is clear that we can build two independent C_2-primary combinations out of the operators in this list. At the moment, we cannot exclude the possibility that there is no constant linear combination of these two C_2-primaries that keeps them orthogonal at finite coupling. Verifying what actually happens would require a perturbative computation at more than 3 loops, which lies beyond our current computational power. Therefore, we cannot currently provide decisive evidence that favors a $\mathrm{U}(1)^4$ holonomy compared to a $\mathrm{U}(1)^2 \times \mathrm{U}(2)$ holonomy in this sector.

4.4 Other implications

The reducibility of the connection has further implications, even for non-extremal correlation functions in the $\mathcal{N}=2$ chiral ring. Consider the general $(n+\overline{n})$-point function in the $\mathcal{N}=2$ chiral ring in the diagonal hatted basis $\hat{\phi}_K$

$$\mathcal{A}_{K_1\cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}} = \langle \hat{\phi}_{K_1}(x_1)\cdots\hat{\phi}_{K_n}(x_n)\overline{\hat{\phi}_{L_1}}(y_1)\cdots\overline{\hat{\phi}_{L_{\overline{n}}}}(y_{\overline{n}})\rangle \tag{4.14}$$

where the total R-charge of the insertions vanishes. The covariant derivative of this correlation function with respect to the complexified gauge coupling τ expresses by definition [2, 4] the renormalized integrated $(n+\overline{n}+1)$-point function

$$\begin{aligned}
\hat{\nabla}_\tau \mathcal{A}_{K_1\cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}} &= \left\langle \int d^4 z\, \mathcal{O}_\tau(z) \hat{\phi}_{K_1}(x_1)\cdots\hat{\phi}_{K_n}(x_n)\overline{\hat{\phi}_{L_1}}(y_1)\cdots\overline{\hat{\phi}_{L_{\overline{n}}}}(y_{\overline{n}})\right\rangle_{\text{renormalized}} \\
&= \partial_\tau \mathcal{A}_{K_1\cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}} - \sum_{i=1}^{n} \left(\hat{A}_\tau\right)_{K_i}^{M} \mathcal{A}_{K_1\cdots K_{i-1} M \cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}} \\
&= \partial_\tau \mathcal{A}_{K_1\cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}} - \left(\sum_{i=1}^{n} \left(\hat{A}_\tau\right)_{K_i}^{K_i}\right) \mathcal{A}_{K_1\cdots K_n \overline{L}_1 \cdots \overline{L}_{\overline{n}}}
\end{aligned} \tag{4.15}$$

where in the last step we assumed the full reducibility of the connection.

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A characteristic example of the general relation (4.15) is the case of the covariant derivative of the 3-point function

$$\hat{C}_{2K\overline{L}} = \hat{C}_{2K}^{M}\hat{g}_{M\overline{L}} = \hat{g}_{K+2,\overline{L}} = \hat{g}_{K+2,\overline{K}+\overline{2}}\,\delta_{\overline{K}+\overline{2},\overline{L}}\,. \tag{4.16}$$

Direct computation of the r.h.s. of equation (4.15) in this case implies the vanishing of the integrated 4-point function

$$\left\langle \int d^4z\,\mathcal{O}_\tau(z)\,\phi_2(x_1)\hat{\phi}_K(x_2)\overline{\hat{\phi}_L}(y)\right\rangle_{\text{renormalized}} = 0\,,\quad \overline{L}\neq\overline{K}+\overline{2}\,. \tag{4.17}$$

Since there is no obvious symmetry reason for this identity, it would be interesting to obtain it with an independent derivation. We suspect that such a derivation might be a useful step towards the ultimate proof of the no-mixing conjecture.

5 Checks in perturbation theory

In this section we compute the first non-trivial quantum corrections to the 2-point functions of chiral primaries in certain examples in SU(N) SCQCD. The first non-trivial correction appears diagrammatically at 3-loops. In all examples we find evidence that the connection on the space of chiral primaries is indeed reducible in accordance with the no-mixing proposal of section 3.3.

More specifically, using the general 3-loop perturbative formula of appendix A, (A.32), we compute the perturbative matrix of 2-point functions up to conformal dimension $\Delta = 8$ for SU(3) and $\Delta = 6$ for SU(4). The explicit computation was performed with Mathematica. We report only these cases at this stage, because as we increase the rank N and the scaling dimension Δ of the operators, the combinatorics of the general formula (A.32) quickly render the computation slow and impractical.

5.1 SU(3) SCQCD up to $\Delta = 8$

We begin with the analysis of 2-point functions in the SU(3) theory. In this case, the $\mathcal{N} = 2$ chiral ring is generated by the chiral primaries

$$\mathrm{Tr}[\varphi^2]\,,\quad \mathrm{Tr}[\varphi^3]\,. \tag{5.1}$$

The first scaling dimension with non-trivial degeneracy is $\Delta = 6$, where we have the operators

$$\mathrm{Tr}[\varphi^2]^3\,,\quad \mathrm{Tr}[\varphi^3]^2\,. \tag{5.2}$$

Notice that in order to determine whether or not the tt^* equations decouple, we need to study 2-point functions up to level 8.

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Applying the formulae and normalization conventions of appendix A, we find the following results

$$G_2 = \left(\frac{g_{YM}^2}{16\pi}\right)^2 \left(16 - \frac{45\,\zeta(3)}{2\pi^4} g_{YM}^4\right), \tag{5.3}$$

$$G_3 = \left(\frac{g_{YM}^2}{16\pi}\right)^3 \left(40 - \frac{135\,\zeta(3)}{2\pi^4} g_{YM}^4\right), \tag{5.4}$$

$$G_4 = \left(\frac{g_{YM}^2}{16\pi}\right)^4 \left(640 - \frac{2160\,\zeta(3)}{\pi^4} g_{YM}^4\right), \tag{5.5}$$

$$G_5 = \left(\frac{g_{YM}^2}{16\pi}\right)^5 \left(1120 - \frac{4410\,\zeta(3)}{\pi^4} g_{YM}^4\right), \tag{5.6}$$

$$G_6 = \left(\frac{g_{YM}^2}{16\pi}\right)^6 \begin{pmatrix} 46080 - \frac{272160\,\zeta(3)}{\pi^4} g_{YM}^4 & 1920 - \frac{11340\,\zeta(3)}{\pi^4} g_{YM}^4 \\ 1920 - \frac{11340\,\zeta(3)}{\pi^4} g_{YM}^4 & 6800 - \frac{57645\,\zeta(3)}{2\pi^4} g_{YM}^4 \end{pmatrix}, \tag{5.7}$$

$$G_7 = \left(\frac{g_{YM}^2}{16\pi}\right)^7 \left(71680 - \frac{483840\,\zeta(3)}{\pi^4} g_{YM}^4\right), \tag{5.8}$$

$$G_8 = \left(\frac{g_{YM}^2}{16\pi}\right)^8 \begin{pmatrix} 5160960 - \frac{46448640\,\zeta(3)}{\pi^4} g_{YM}^4 & 215040 - \frac{1935360\,\zeta(3)}{\pi^4} g_{YM}^4 \\ 215040 - \frac{1935360\,\zeta(3)}{\pi^4} g_{YM}^4 & 277760 - \frac{2046240\,\zeta(3)}{\pi^4} g_{YM}^4 \end{pmatrix}. \tag{5.9}$$

The 2×2 matrix G_6 is written in the basis $\mathrm{Tr}[\varphi^2]^3, \mathrm{Tr}[\varphi^3]^2$, while G_8 is written in the basis $\mathrm{Tr}[\varphi^2]^4, \mathrm{Tr}[\varphi^2]\mathrm{Tr}[\varphi^3]^2$. It is manifest that this basis does not diagonalize the 2-point functions, not even at tree-level. As explained in previous sections, we can diagonalize the 2-point functions by constructing the C_2-primaries. $\mathrm{Tr}[\varphi^3]^2$, in particular, is not a C_2-primary, as can be easily seen from the tree-level OPEs

$$\mathrm{Tr}[\overline{\varphi}^2](x)\mathrm{Tr}[\varphi^2]^3(0) \approx \frac{9g_{YM}^4}{32\pi^2|x|^4}\mathrm{Tr}[\varphi^2]^2(0) + \dots, \tag{5.10}$$

$$\mathrm{Tr}[\overline{\varphi}^2](x)\mathrm{Tr}[\varphi^3]^2(0) \approx \frac{3g_{YM}^4}{256\pi^2|x|^4}\mathrm{Tr}[\varphi^2]^2(0) + \dots. \tag{5.11}$$

It is then easy to take an appropriate linear combination of the two chiral primaries of dimension 6 that is annihilated by $C_2^\dagger$. The appropriate bases at scaling dimensions 6 and 8 are then given by the operators

$$\phi_6 = \mathrm{Tr}[\varphi^2]^3, \qquad \phi_{6'} = \mathrm{Tr}[\varphi^3]^2 - \frac{1}{24}\mathrm{Tr}[\varphi^2]^3, \tag{5.12}$$

$$\phi_8 = \phi_2\phi_6, \qquad \phi_{8'} = \phi_2\phi_{6'}, \tag{5.13}$$

where it is easily checked that $C_2^\dagger \cdot \phi_{6'} = 0$. In the new basis, the 2-point functions become diagonal even when we include the first non-trivial quantum corrections

$$G'_6 = \left(\frac{g_{YM}^2}{16\pi}\right)^6 \begin{pmatrix} 46080 - \frac{272160\,\zeta(3)}{\pi^4} g_{YM}^4 & 0 \\ 0 & 6720 - \frac{28350\,\zeta(3)}{\pi^4} g_{YM}^4 \end{pmatrix}, \tag{5.14}$$

$$G'_8 = \left(\frac{g_{YM}^2}{16\pi}\right)^8 \begin{pmatrix} 5160960 - \frac{46448640\,\zeta(3)}{\pi^4} g_{YM}^4 & 0 \\ 0 & 268800 - \frac{1965600\,\zeta(3)}{\pi^4} g_{YM}^4 \end{pmatrix} \tag{5.15}$$

verifying at this order the no-mixing conjecture.

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It is also easy to check that the correlators satisfy the appropriate Toda chains (3.20), as explained in the previous sections.

5.2 SU(4) SCQCD up to $\Delta = 6$

In this section we study correlation functions in the SU(4) theory. In this case the $\mathcal{N} = 2$ chiral ring is generated by the three chiral primaries

$$\mathrm{Tr}[\varphi^2]\,, \quad \mathrm{Tr}[\varphi^3]\,, \quad \mathrm{Tr}[\varphi^4]\,. \tag{5.16}$$

Consequently, the spectrum is already degenerate at $\Delta = 4$, where we have the two degenerate operators

$$\mathrm{Tr}[\varphi^2]^2\,, \quad \mathrm{Tr}[\varphi^4]\,. \tag{5.17}$$

At $\Delta = 6$, we have an additional degeneracy compared to the SU(3) case, as we have the three independent operators

$$\mathrm{Tr}[\varphi^2]^3\,, \quad \mathrm{Tr}[\varphi^2]\mathrm{Tr}[\varphi^4]\,, \quad \mathrm{Tr}[\varphi^3]\mathrm{Tr}[\varphi^3]\,. \tag{5.18}$$

Applying the formulae of appendix A, we find the 2-point functions

$$G_2 = \left(\frac{g_{YM}^2}{16\pi}\right)^2 \left(30 - \frac{2295\,\zeta(3)}{32\pi^4} g_{YM}^4\right)\,, \tag{5.19}$$

$$G_3 = \left(\frac{g_{YM}^2}{16\pi}\right)^3 \left(135 - \frac{23085\,\zeta(3)}{64\pi^4} g_{YM}^4\right)\,, \tag{5.20}$$

$$G_4 = \begin{pmatrix} 2040 - \frac{43605\,\zeta(3)}{4\pi^4} g_{YM}^4 & 870 - \frac{74385\,\zeta(3)}{16\pi^4} g_{YM}^4 \\ 870 - \frac{74385\,\zeta(3)}{16\pi^4} g_{YM}^4 & \frac{1335}{2} - \frac{198045\,\zeta(3)}{64\pi^4} g_{YM}^4 \end{pmatrix}\,, \tag{5.21}$$

$$G_5 = \left(\frac{g_{YM}^2}{16\pi}\right)^5 \left(5670 - \frac{535815\,\zeta(3)}{16\pi^4} g_{YM}^4\right)\,, \tag{5.22}$$

$$G_6 = \left(\frac{g_{YM}^2}{16\pi}\right)^6 \begin{pmatrix} 232560 - \frac{82241345\zeta(3)}{4\pi^4} g_{YM}^4 & 99180 - \frac{14058765\zeta(3)}{16\pi^4} g_{YM}^4 & 6480 - \frac{229635\zeta(3)}{4\pi^4} g_{YM}^4 \\ 99180 - \frac{14058765\zeta(3)}{16\pi^4} g_{YM}^4 & 55935 - \frac{30324105\zeta(3)}{64\pi^4} g_{YM}^4 & 8100 - \frac{1012095\zeta(3)}{16\pi^4} g_{YM}^4 \\ 6480 - \frac{229635\zeta(3)}{4\pi^4} g_{YM}^4 & 8100 - \frac{1012095\zeta(3)}{16\pi^4} g_{YM}^4 & 58320 - \frac{1454355\zeta(3)}{4\pi^4} g_{YM}^4 \end{pmatrix}. \tag{5.23}$$

As before, we can find a constant linear rotation that diagonalizes the 2-point functions at tree level by finding the appropriate C_2-primary combinations. The new basis is given by the operators

$$\phi_4 = \mathrm{Tr}[\varphi^2]^2\,, \qquad \phi_{4'} = \mathrm{Tr}[\varphi^4] - \frac{29}{68}\mathrm{Tr}[\varphi^2]^2\,, \tag{5.24}$$

$$\phi_6 = \phi_2\phi_4\,, \quad \phi_{6'} = \phi_2\phi_{4'}\,, \quad \phi_{6''} = \mathrm{Tr}[\varphi^6] - \frac{9}{23}\mathrm{Tr}[\varphi^2]\mathrm{Tr}[\varphi^4] + \frac{243}{1748}\mathrm{Tr}[\varphi^2]^3 \tag{5.25}$$

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and the corresponding 2-point functions are given by

$$G'_4 = \left(\frac{g_{YM}^2}{16\pi}\right)^4 \begin{pmatrix} 2040 - \frac{43605\,\zeta(3)}{4\pi^4} g_{YM}^4 & 0 \\ 0 & \frac{5040}{17} - \frac{18900\,\zeta(3)}{17\pi^4} g_{YM}^4 \end{pmatrix}, \tag{5.26}$$

$$G'_6 = \left(\frac{g_{YM}^2}{16\pi}\right)^8 \begin{pmatrix} 232560 - \frac{8241345\,\zeta(3)}{4\pi^4} g_{YM}^4 & 0 & 0 \\ 0 & \frac{231840}{17} - \frac{3368925\,\zeta(3)}{34\pi^4} g_{YM}^4 & 0 \\ 0 & 0 & \frac{24494400}{437} - \frac{151559100\,\zeta(3)}{437\pi^4} g_{YM}^4 \end{pmatrix}, \tag{5.27}$$

Once again the no-mixing conjecture is verified.

It is also easy to verify that the diagonal components of the above 2-point functions obey (3.20).

6 Outlook

The observations in this paper suggest the existence of a new interesting class of non-renormalization theorems in four-dimensional $\mathcal{N} = 2$ superconformal field theories. It would be important to prove these theorems in the SU(N) $\mathcal{N} = 2$ SCQCD theories, and to clarify whether the holonomy of the chiral primary vector bundles is fully or partially reducible.

We emphasized that full reducibility is a consistent ansatz from the point of view of the tt^* equations, which reduces them to an independent set of semi-infinite Toda chains. The non-perturbative solution of the 2-point functions in each of these chains requires a single external datum. It would be interesting to explore techniques that will allow the exact computation of these data generalizing the success of supersymmetric localization on the four-sphere for the Zamolodchikov metric.

We would also like to highlight the efficiency of our results already at tree level. The tree level formulae derived in this paper are also applicable in the same form in the context of chiral primaries in $\mathcal{N} = 4$ SYM theory.

In conclusion, in this paper we have seen that the study of the tt^* equations is a powerful guide towards new exact results in four-dimensional quantum field theories. It would be extremely interesting to study the solution of the tt^* equations in more general classes of $\mathcal{N} = 2$ superconformal field theories, and to examine the possibility of more general non-renormalization theorems in $\mathcal{N} = 2$ theories. At face value, the appearance of such theorems in $\mathcal{N} = 2$ theories is rather unexpected. Perhaps there are similar surprises in $\mathcal{N} = 1$ theories as well. It would be interesting to explore this possibility.

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A Perturbative 2- and 3-point functions in SU(N) $\mathcal{N} = 2$ SCQCD theory

In this appendix we summarize the details of a perturbative computation that determines the general 2-point function in the $\mathcal{N} = 2$ chiral ring of SU(N) $\mathcal{N} = 2$ SCQCD theory up to 3 loops. Since the OPE coefficients are completely fixed in our conventions, our computation also gives results for the perturbative form of the general 3-point functions in the $\mathcal{N} = 2$ chiral ring. As explained in the main text, the $\mathcal{N} = 2$ chiral primaries of interest are general multi-trace operators of the form

$$\phi_{\{n_s\}} = \mathcal{N}_{\{n_s\}} \prod_{s=1}^{N-1} \left(\mathrm{Tr}\left[\varphi^{s+1}\right]\right)^{n_s} \tag{A.1}$$

where $\mathcal{N}_{\{n_s\}}$ are constant normalization factors that will be fixed shortly, and φ is the adjoint complex scalar field in the $\mathcal{N} = 2$ vector multiplet.

By convention, we consider the trace in the fundamental representation of the SU(N) gauge group and normalize the Lie algebra generators T_a $(a = 1, 2, \ldots, N^2 - 1)$ so that

$$\mathrm{Tr}\left[T_a T_b\right] = \delta_{ab} \,. \tag{A.2}$$

The fully antisymmetric symbol f_{abc}, and the fully symmetric symbol d_{abc} are defined as usual

$$f_{abc} = -i\, \mathrm{Tr}\left[[T_a, T_b]T_c\right] \,, \quad d_{abc} = \mathrm{Tr}\left[\{T_a, T_b\}T_c\right] \,. \tag{A.3}$$

Then, expressing the adjoint complex scalar field φ as

$$\varphi = \varphi^a T_a \tag{A.4}$$

we can recast the generic chiral primary (A.1) of scaling dimension Δ into the form

$$\phi_{\{n_s\}} = \mathcal{N}_{\{n_s\}}\, \mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\, \varphi^{a_1} \cdots \varphi^{a_\Delta} \,, \quad \Delta = \sum_{s=1}^{N-1} (s+1)\, n_s \,, \tag{A.5}$$

where

$$\begin{aligned} \mathcal{C}_{\{n_s\};a_1\cdots a_\Delta} = &\left(\mathrm{Tr}\left[T_{a_1} T_{a_2}\right] \cdots \mathrm{Tr}\left[T_{a_{2n_1-1}} T_{a_{2n_1}}\right]\right) \\ &\left(\mathrm{Tr}\left[T_{a_{2n_1+1}} T_{a_{2n_1+2}} T_{a_{2n_1+3}}\right] \cdots \mathrm{Tr}\left[T_{a_{2n_1+3n_2-2}} T_{a_{2n_1+3n_2-1}} T_{a_{2n_1+3n_2}}\right]\right) \cdots \end{aligned} \tag{A.6}$$

is the obvious product of traces of Lie algebra generators.

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Following the conventions of ref. [6] at tree level the 2-point function of the adjoint scalar components φ^a is

$$\left\langle \varphi^a(x)\,\overline{\varphi}^b(0)\right\rangle = \delta^{ab}\frac{1}{\pi\,\mathrm{Im}\tau}\frac{1}{|x|^2}\,. \tag{A.7}$$

We fix the constant normalization factors $\mathcal{N}_{\{n_s\}}$ of the operators $\phi_{\{n_s\}}$ so that (after the standard Wick contractions) these operators have tree-level 2-point functions

$$\left\langle \phi_{\{n_s\}}(x)\overline{\phi}_{\{n_s\}}(0)\right\rangle = \frac{1}{(4\,\mathrm{Im}\tau)^{\Delta}}\,\mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\sum_{\sigma\in\mathcal{S}_\Delta}\mathcal{C}_{\{n_s\};a_{\sigma(1)}\cdots a_{\sigma(\Delta)}}\frac{1}{|x|^{2\Delta}}\,, \tag{A.8}$$

where $\mathcal{S}_\Delta$ is the permutation group of Δ elements. This choice is consistent with the normalization that leads to the tt^* equations (1.3); in particular, it is consistent with the OPE

$$\phi_{\{n_s\}}\cdot\phi_{\{m_s\}}\sim\phi_{\{n_s+m_s\}}\,. \tag{A.9}$$

It is convenient to compute perturbative corrections to correlation functions in the $\mathcal{N}=2$ SCQCD theory using supergraph methods in $\mathcal{N}=1$ superspace language. In fact, the relevant computation of 2-point functions in the $\mathcal{N}=2$ chiral ring up to order $\mathcal{O}(g^4_{YM})$ in the Yang-Mills coupling g is quite similar to a 3-loop computation of 2-point functions of chiral primary operators in $\mathcal{N}=4$ SYM theory performed previously in [24]. As expected by the known non-renormalization theorems, and verified explicitly in [24], the correction in $\mathcal{N}=4$ SYM theory vanishes. Hence, it is convenient to perform the $\mathcal{N}=2$ SCQCD computation by subtracting the corresponding contributions of the analogous computation in $\mathcal{N}=4$ SYM theory (the same approach in this context was employed successfully in the past using standard Feynman diagrams in real space in [6, 25]).

In $\mathcal{N}=1$ superspace language the 2-point functions of interest take the form

$$\left\langle \phi_{\{n_s\}}(z_1)\,\overline{\phi}_{\{\overline{n}_s\}}(z_2)\right\rangle = \frac{F(\{n_s\},\{\overline{n}_s\},g^2_{YM})}{(x_1-x_2)^{2\Delta}}\delta^{(4)}(\theta_1-\theta_2) \tag{A.10}$$

where $z=(x,\theta,\overline{\theta})$ are superspace coordinates. We are after the perturbative form of the spacetime-independent factor F

$$F = F_0 + g^2_{YM}F_2 + g^4_{YM}F_4 + \mathcal{O}(g^6_{YM})\,. \tag{A.11}$$

In the SU(N) $\mathcal{N}=2$ SCQCD theory besides the $\mathcal{N}=1$ vector superfield V and the adjoint chiral superfield φ we have $N_f=2N$ fundamental doublets of chiral superfields $Q_i,\tilde{Q}_i$. Following closely the superspace conventions of ref. [24] (with the obvious additional features of $\mathcal{N}=2$ SCQCD compared to $\mathcal{N}=4$ SYM) we have four types of super-propagators

V propagator : (A.12)

φ propagator : (A.13)

Q_i propagator : (A.14)

$\tilde{Q}_i$ propagator : (A.15)

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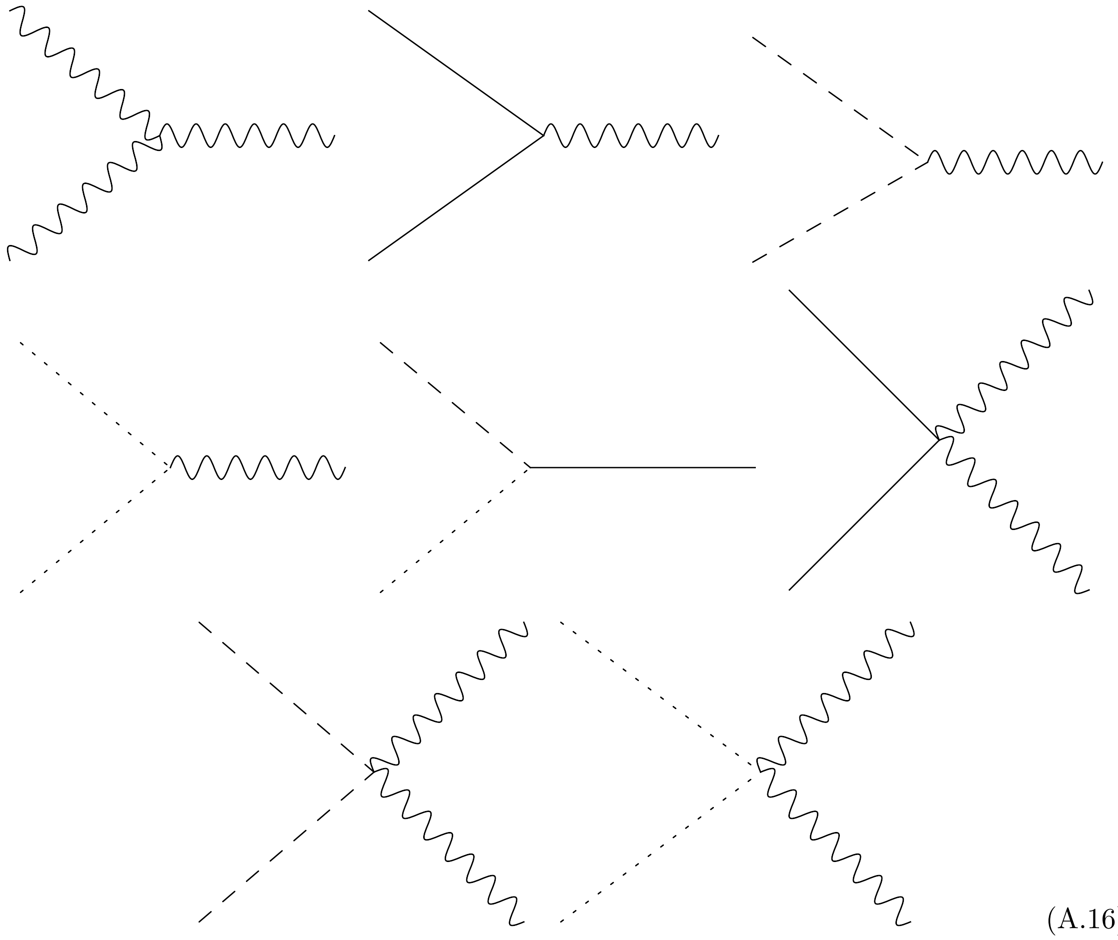
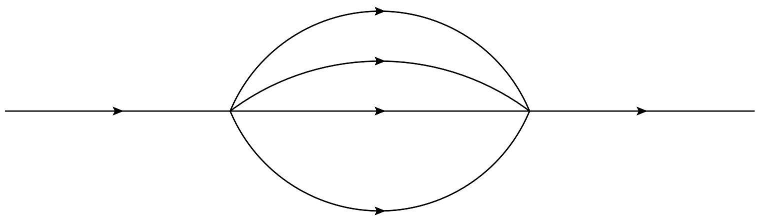
There are also eight types of super-vertices



(A.16)

We will now sketch how different contributions to the function F arise up to 3 loops in perturbation theory highlighting the differences from the $\mathcal{N}=4$ SYM case (a detailed exposition of several needed facts can be found in ref. [24]).

At tree level the non-color factor is evaluated from the super-Feynman diagram



(A.17)

as in $\mathcal{N}=4$ SYM theory. In our conventions the result is

$$F_0 = \frac{1}{(4\,\mathrm{Im}\tau)^{\Delta}}\,\mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\sum_{\sigma\in\mathcal{S}_\Delta}\mathcal{C}_{\{\overline{n}_s\};a_{\sigma(1)}\cdots a_{\sigma(\Delta)}} \tag{A.18}$$

in agreement with equation (A.8).

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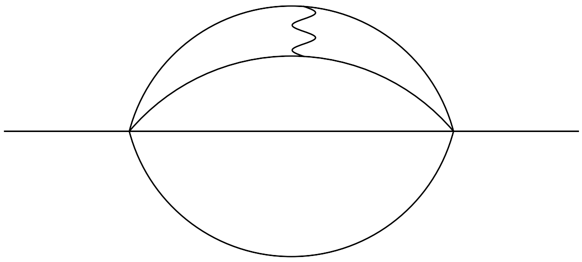
At the next order, $\mathcal{O}(g_{YM}^2)$, the only potential contribution comes from diagrams of the form



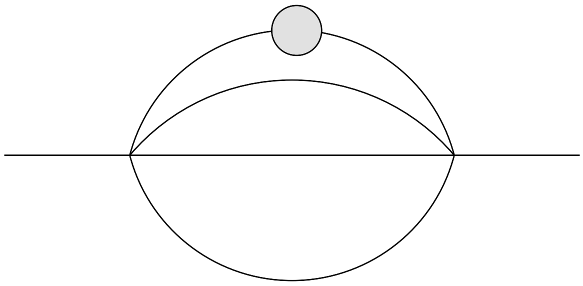
(A.19)

However, as explained in [24] none of these diagrams give a requisite $1/\varepsilon$ pole in dimensional regularization, and as a result, there is no contribution to F_2. Namely,

$$F_2 = 0\,. \tag{A.20}$$

The first non-vanishing correction arises at order $\mathcal{O}(g_{YM}^4)$. Besides the diagrams that are common with $\mathcal{N}=4$ SYM theory (and will not be listed here) the contributing diagrams to the non-color factor in $\mathcal{N}=2$ SCQCD theory at this order are

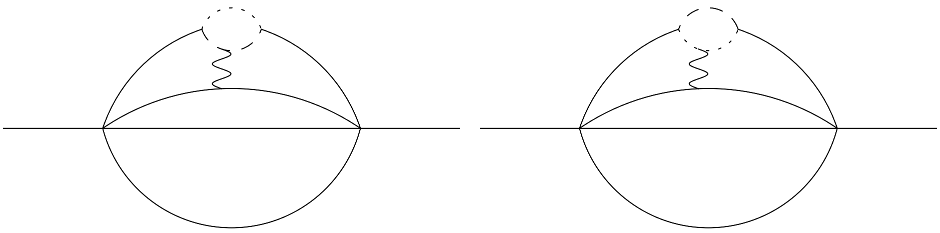
(1)



(A.21)

that involves the 2-loop corrected φ-propagator,

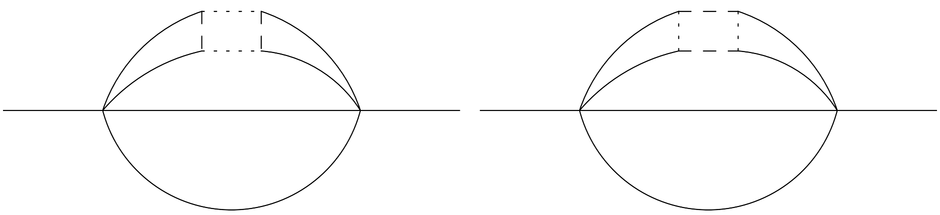
(2)



(A.22)

that correct the effective $\overline{\varphi}\varphi V$ vertex,

(3)



(A.23)

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(4)

(A.24)

The diagrams (3), like the diagrams $3f, 3g, 3h$ in [24] do not contribute to the 2-point functions. Hence, summing the contributions of the diagrams (1), (2), (4) we get

$$F_4 = F_4^{(1)} + F_4^{(2)} + F_4^{(4)}\,. \tag{A.25}$$

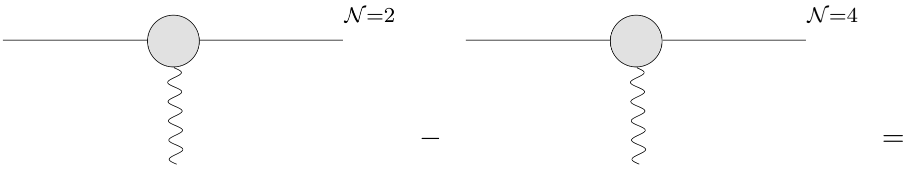
The difference $(\mathcal{N}=2)-(\mathcal{N}=4)$ between the $\mathcal{N}=2$ and $\mathcal{N}=4$ results for the 2-loop corrected propagator is [25][3]

$$12\;\zeta(3)\,g_{YM}^4\,(N^2+1)\,\frac{1}{(p^2)^{2\varepsilon}}\,. \tag{A.26}$$

Then, performing the combinatorics and the D-algebra of the full diagram precisely as in [24] we obtain

$$F_4^{(1)} = -\left(\frac{1}{4\pi}\right)^4\left(\frac{1}{4\,\mathrm{Im}\tau}\right)^{\Delta}12\Delta(N^2+1)\,\zeta(3)\;\mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\sum_{\sigma\in\mathcal{S}_\Delta}\mathcal{C}_{\{\overline{n}_s\};a_{\sigma(1)}\cdots a_{\sigma(\Delta)}}\,. \tag{A.27}$$

Similarly, we can easily deduce the $(\mathcal{N}=2)-(\mathcal{N}=4)$ difference for the effective $\overline{\varphi}\varphi V$ vertex



(A.28)

$$\frac{Ng^3}{4}d_{abc}\overline{\varphi}^a(q,\theta)\varphi^a(-p,\theta)\Big(4D^\alpha\overline{D}^2D_\alpha+(p+q)^{\alpha\dot\alpha}\left[D_\alpha,\overline{D}_{\dot\alpha}\right]\Big)V^c(p-q,\theta)\int\frac{d^nk}{k^2(k-p)^2(k-q)^2}\,.$$

Doing the full D-algebra as in [24] we finally obtain

$$F_4^{(2)} = \left(\frac{1}{4\pi}\right)^4\left(\frac{1}{4\mathrm{Im}\tau}\right)^{\Delta}12N\zeta(3)\mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\sum_\sigma\sum_{j\neq\ell}\mathcal{C}_{\{\overline{n}_s\};a_{\sigma(1)}\cdots b_j\cdots b_\ell\cdots a_{\sigma(\Delta)}}\,if_{a_{\sigma(j)}mb_j}d_{a_{\sigma(\ell)}mb_\ell}. \tag{A.29}$$

For the final term $F_4^{(4)}$ we compute only the contribution of the diagrams (4) and subtracting the contribution of the corresponding $\mathcal{N}=4$ diagrams with the adjoint superfields running in the loop we find

$$F_4^{(4)} = \left(\frac{1}{4\pi}\right)^4\left(\frac{1}{4\,\mathrm{Im}\tau}\right)^{\Delta}12\,\zeta(3)\;\mathcal{C}_{\{n_s\};a_1\cdots a_\Delta}\sum_\sigma\sum_{j\neq\ell}\mathcal{C}_{\{\overline{n}_s\};a_{\sigma(1)}\cdots b_j\cdots b_\ell\cdots a_{\sigma(\Delta)}}\mathcal{D}_{b_ja_{\sigma(j)}b_\ell a_{\sigma(\ell)}} \tag{A.30}$$

[3]Note that compared to equation (18) of [25] in our Lie algebra conventions the r.h.s. of the equation is $2(N^2+1)$ versus their $\frac{N^2+1}{2}$.

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where

$$\mathcal{D}_{abcd} = \frac{N}{2} \mathrm{Tr}\,[T_a T_b T_c T_d] - \frac{1}{4} f_{amn} f_{npd} f_{cpq} f_{qmb}\,. \tag{A.31}$$

We are now in position to collect the final result for the perturbative correction at 3 loops[4]

$$F_4 = \left(\frac{1}{4\pi}\right)^4 \left(\frac{1}{4\mathrm{Im}\tau}\right)^\Delta 12\,\zeta(3)\; \mathcal{C}_{\{n_s\};a_1\cdots a_\Delta} \tag{A.32}$$

$$\sum_\sigma \left[-(N^2+1)\Delta \mathcal{C}_{\{\overline{n}_s\};a_{\sigma(1)}\cdots a_{\sigma(\Delta)}} + \sum_{j\neq \ell} \overline{\mathcal{C}}_{a_{\sigma(1)}\cdots b_j \cdots b_\ell \cdots a_{\sigma(k)}} \left(iN f_{a_{\sigma(j)} m b_j} d_{a_{\sigma(\ell)} m b_\ell} + \mathcal{D}_{b_j a_{\sigma(j)} b_\ell a_{\sigma(\ell)}}\right)\right].$$

As a check of these results we have verified that the above formula for $F = F_0 + g_{YM}^4 F_4$ reproduces correctly the Zamolodchikov metric [7]

$$g_2 \equiv \left(\frac{\pi}{4}\right)^2 \left\langle \mathrm{Tr}[\varphi^2](1)\mathrm{Tr}[\overline{\varphi}^2](0)\right\rangle = \partial_\tau \partial_{\overline{\tau}} \log Z_{S^4} \tag{A.33}$$

in the case of the gauge groups SU(2), SU(3), and SU(4), when the exact S^4 partition function (determined by localization [9]) is expanded at this order.

B Explicit diagonalization of 2-point functions

Diagonalization of 2-point functions. The diagonalization of the matrix of 2-point function coefficients $g_{K\overline{L}}$ can be performed in different ways. Gram-Schmidt diagonalization is a standard prescription where one picks a first vector ϕ_{K_1}, then combines it with a second vector ϕ_{K_2} to find a linear combination orthogonal to ϕ_{K_1}, then combines ϕ_{K_1} and ϕ_{K_2} with a third vector ϕ_{K_3} to find a linear combination orthogonal to the previous two orthogonal vectors and so on and so forth. The choice of the order of the vectors $\phi_{K_1},\ldots$ in this prescription translates to different linear transformations between the original and the orthogonal bases.

In what follows, we adopt a slight variant of the Gram-Schmidt diagonalization procedure that reproduces the results of section 3.1 based on the C_2-algebra (3.16). We single out the first vector ϕ_{K_1} in the multi-trace basis (2.2) as a chiral primary operator with the maximum number of ϕ_2 factors. Then, we perform a first linear transformation

$$(\phi^{(1)})_K = (\mathcal{M}_1)_K{}^L\, \phi_L \tag{B.1}$$

that leaves ϕ_{K_1} unchanged and transforms all the remaining vectors to set $\langle \phi_L^{(1)} \overline{\phi}_{K_1}\rangle = 0$ for $L \neq K_1$. A general matrix $(\mathcal{M}_1)_K{}^L$ with these properties takes the form

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -\sum_{L\neq K_1} \mu_{L_1\overline{L}} \frac{g_{L\overline{K}_1}}{g_{K_1\overline{K}_1}} & \mu_{L_1\overline{L}_1} & \mu_{L_1\overline{L}_2} & \cdots \\ -\sum_{L\neq K_1} \mu_{L_2\overline{L}} \frac{g_{L\overline{K}_1}}{g_{K_1\overline{K}_1}} & \mu_{L_2\overline{L}_1} & \mu_{L_2\overline{L}_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{B.2}$$

[4]In all explicit SU(3) and SU(4) examples that we worked out the term proportional to the symmetric symbol d_{abc} was found not to contribute at the end. It is interesting to examine if this is a generic property.

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where the indices L_i refer to chiral primaries other than ϕ_{K_1} and the matrix elements $\mu_{L_i\overline{L}_j}$ are free for the moment. We will generate non-trivial entries $\mu_{L_i\overline{L}_j}$ sequentially, hence at this stage we adopt a scheme where $\mu_{L_i\overline{L}_j} = \delta_{L_i\overline{L}_j}$. Then,

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ -\frac{g_{L_1\overline{K}_1}}{g_{K_1\overline{K}_1}} & 1 & 0 & \cdots \\ -\frac{g_{L_2\overline{K}_1}}{g_{K_1\overline{K}_1}} & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} . \tag{B.3}$$

At the second step we single out a vector $\phi^{(1)}_{K_2}$ (other than ϕ_{K_1}), with the next largest number of ϕ_2 factors, and repeat the same transformation in the subspace that excludes ϕ_{K_1}. Accordingly, we perform a second linear transformation

$$(\phi^{(2)})_K = (\mathcal{M}_2)_K{}^L\, (\phi^{(1)})_L \tag{B.4}$$

with

$$\mathcal{M}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & -\frac{g^{(1)}_{L_1\overline{K}_2}}{g^{(1)}_{K_2\overline{K}_2}} & 1 & 0 & \cdots \\ 0 & -\frac{g^{(1)}_{L_2\overline{K}_2}}{g^{(1)}_{K_2\overline{K}_2}} & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} . \tag{B.5}$$

By $g^{(1)}_{K\overline{L}}$ we have denoted the 2-point function coefficients in the transformed basis $\phi^{(1)}_K$,

$$g^{(1)}_{K\overline{L}} = g_{K\overline{L}} - \frac{g_{K\overline{K}_1} g_{K_1\overline{L}}}{g_{K_1\overline{K}_1}} \tag{B.6}$$

for $K, L \neq K_1$.

We continue in this fashion until the full diagonalization of the matrix $g_{K\overline{L}}$. The complete transformation matrix is

$$\mathcal{M} = \mathcal{M}_{D_R-1} \ldots \mathcal{M}_2 \mathcal{M}_1 \tag{B.7}$$

where D_R is the degeneracy of the chiral primary fields with $\mathrm{U}(1)_R$ charge R. The chiral primaries in the new basis are

$$\hat{\phi}_K = \mathcal{M}_K{}^L \phi_L \tag{B.8}$$

and the matrix of 2-point function coefficients is diagonal

$$\langle \hat{\phi}_K \overline{\hat{\phi}}_L \rangle = \hat{g}_{K\overline{L}} = \hat{g}_{K\overline{K}} \delta_{\overline{KL}} . \tag{B.9}$$

We encounter the same freedom in this process that we encountered also in section 3.1. When two operators have the same number of ϕ_2 factors it is unclear which order we should proceed in.

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The no-mixing conjecture of section 3.3 postulates that the new section remains holomorphic, i.e. the linear transformation matrix $\mathcal{M}$ is a holomorphic function of the moduli

$$\partial_{\overline{\tau}} \mathcal{M} = 0 \,. \tag{B.10}$$

This is equivalent to the conditions

$$\partial_{\overline{\tau}} \mathcal{M}_i = 0 \,, \quad i = 1, 2, \ldots, D_R - 1 \,, \tag{B.11}$$

which translate to

$$\partial_{\overline{\tau}} \left(\frac{g^{(i-1)}_{L\overline{K}_i}}{g^{(i-1)}_{K_i\overline{K}_i}} \right) = 0 \,, \quad i = 1, 2, \ldots, D_R - 1 \,. \tag{B.12}$$

By definition $g^{(0)}_{K\overline{L}} = g_{K\overline{L}}$.

As expected by consistency, these relations are invariant under a holomorphic rescaling of the chiral primary fields. Moreover, they imply that by suitable holomorphic rescalings it is possible to adopt a more specific set of normalization conventions where all the 2-point function coefficients $g_{K\overline{L}}$ are real. This is the real ϕ_K basis that was aluded to in the main text and was explicit in the perturbative computations. In this basis the complex conjugate of the relations (B.12) implies that the ratio of 2-point functions is also τ-independent. As a result, in the real basis the ratios $\frac{g^{(i-1)}_{L\overline{K}_i}}{g^{(i-1)}_{K_i\overline{K}_i}}$ are coupling constant independent and their value is fixed at tree level, namely

$$\frac{g^{(i-1)}_{L\overline{K}_i}}{g^{(i-1)}_{K_i\overline{K}_i}} = \left(\frac{g^{(i-1)}_{L\overline{K}_i}}{g^{(i-1)}_{K_i\overline{K}_i}} \right)_{\text{tree-level}} \,. \tag{B.13}$$

This equation is a statement of non-renormalization formulated in a local patch (based on the holomophic gauge) on the superconformal manifold of the $\mathcal{N} = 2$ SCQCD theory.

Diagonalization of C_2. In the construction of section 3.1 based on the C_2-algebra (3.16) the simultaneous diagonalization of the OPE coefficient C_2 was automatic. In the above language this property can be formulated as follows. In the original basis (2.2) the normalization conventions guarantee $C^L_{2K} = \delta^L_{K+2}$. After the linear transformation (B.7) we obtain the new OPE coefficients

$$\hat{C}^L_{2K} = (\mathcal{M}_{(\Delta)})_K{}^S \, C^P_{2S} \, (\mathcal{M}^{-1}_{(\Delta+2)})_P{}^L = (\mathcal{M}_{(\Delta)})_K{}^S (\mathcal{M}^{-1}_{(\Delta+2)})_{S+2}{}^L \,, \tag{B.14}$$

where we used the fact that the chiral primary $\phi_2 \propto \mathrm{Tr}[\phi^2]$ is the single scaling dimension 2 operator and does not transform. Also, we used the notation $\mathcal{M}_{(\Delta)}$ to denote the tranformation matrix at scaling dimension Δ. Notice that transformation matrices at two different scaling dimensions appear on the r.h.s. of equation (B.14). It is obvious that the dimensionality of the transformation matrices remains the same or increases as the scaling dimension increases, i.e. $D_{2\Delta} \leq D_{2(\Delta+2)}$.

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Let us phrase the precise conditions that guarantee that the transformed OPE coefficients $\hat{C}^L_{2K}$ remain proportional to δ^L_{K+2}. Part of our prescription above is to organize the rows and columns of the transformation matrix $\mathcal{M}_{(\Delta+2)}$ so that its i-th row and column (for $i \leq D_{2\Delta}$) refers to the chiral primary of the i-th row and column of $\mathcal{M}_\Delta$ after the OPE with ϕ_2. It is then straightforward to verify that

$$\hat{C}^L_{2K} = \delta^L_{K+2} \quad \Leftrightarrow \quad \frac{g^{(i-1)}_{L\overline{K}_i}}{g^{(i-1)}_{K_i\overline{K}_i}} = \frac{g^{(i-1)}_{L+2\ \overline{K}_i+\overline{2}}}{g^{(i-1)}_{K_i+2\ \overline{K}_i+\overline{2}}}\,, \quad i = 1, 2, \dots, D_R\,. \tag{B.15}$$

The above relations from the viewpoint of the tt^* equations before the transformation. Before we end this appendix, we would like to present a slighlty different description of the above relations from the point of the view of the tt^* equations in the original basis (2.1). Returning to the tt^* equations (1.3) in the multi-trace basis (2.2) we single out the first chiral primary ϕ_{K_1} (that takes part in the above diagonalization procedure, see eq. (B.1)), and consider the component of the equations with $K = K_1$ and $L \neq K_1$

$$\partial_{\overline{\tau}}\left(g^{\overline{M}L}\partial_\tau g_{K_1\overline{M}}\right) = g_{K_1+2,\overline{R}+\overline{2}}\, g^{\overline{R}L} - g_{K_1\overline{R}}\, g^{\overline{R}-\overline{2},L-2}\,. \tag{B.16}$$

It follows easily from the previous discussion that the non-renormalization equations (B.13) set the l.h.s. (connection part) of this equation to zero. The r.h.s. vanishes as a consequence of equations (B.15). Indeed, using these equations

$$\frac{g_{K_1+2,\overline{M}+\overline{2}}}{g_{K_1+2,\overline{K}_1+\overline{2}}} = \frac{g_{K_1\overline{M}}}{g_{K_1\overline{K}_1}} = \frac{g_{K_1-2,\overline{M}-\overline{2}}}{g_{K_1-2,\overline{K}_1-\overline{2}}} \tag{B.17}$$

and we can recast the r.h.s. in the form

$$\text{r.h.s.} \;= \frac{g_{K_1+2,\overline{K}_1+\overline{2}}}{g_{K_1\overline{K}_1}} g_{K_1\overline{R}}\, g^{\overline{R}L} - \frac{g_{K_1,\overline{K}_1}}{g_{K_1-2,\overline{K}_1-\overline{2}}} g_{K_1-2,\overline{R}-\overline{2}}\, g^{\overline{R}-\overline{2},L-2} = 0 \tag{B.18}$$

since $g_{K_1\overline{R}}g^{\overline{R}L} = 0$, $g_{K_1-2,\overline{R}-\overline{2}}g^{\overline{R}-2,L-2} = 0$.

To proceed with the remaining tt^* equations, one can perform the transformation (B.1), decouple the chiral primary ϕ_{K_1}, repeat the same argument for $\phi^{(1)}_{K_2}$ with the remaining tt^* equations, and so on and so forth.

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